Note di Matematica
Note Mat. 1(2008), suppl. n. 1, 209-232
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1supplp209
http://siba2.unile.it/notemat

2-harmonic maps and their first and second variational formulas [i]

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Received: 22/8/07; accepted: 5/10/07.

Abstract. In [1], J. Eells and L. Lemaire introduced the notion of a k-harmonic map. In this paper we study the case $k = 2$, derive the first and second variational formulas of the 2-harmonic maps, give nontrivial examples of 2-harmonic maps and give proofs of nonexistence theorems of stable 2-harmonic maps.

Keywords: harmonic maps, biharmonic maps, second variation formula

MSC 2000 classification: primary 58E20, secondary 53C43

Introduction

As well known, harmonic maps between Riemannian manifolds $f : M \to N$, where M is compact, can be considered as critical maps of the energy functional $E(f) = \int_M \|df\|^2 *1$. Considering the similar ideas, in 1981, J. Eells and L. Lemaire [1], proposed the problem to consider the k-harmonic maps: critical maps of the functional

$$E_k(f) = \int_M \|(d + d^*)^k f\|^2 *1.$$

In this paper, we consider the case $k = 2$ and show the preliminary results.

We use mainly vector bundle valued differential forms and Riemannian metrics. In §1, we prepare the notation and fundamental formulas needed in the sequel.

In §2, given a compact manifold M, we derive the first variation formula of $E_2(f) = \int_M \|(d+d^*)^2 f\|^2 *1$ (Theorem 1) and give the definition of 2-harmonic maps $f : M \to N$ whose tension field $\tau(f)$ satisfies

$$-\overline{\nabla}^*\overline{\nabla}\tau(f) + R^N(df(e_k), \tau(f))df(e_k) = 0,$$

namely $\tau(f)$ is a solution of the Jacobi type equation.

[i]Reprinted from **Chinese Ann. Math., Ser. A, 7(1986), 389-402**, upon formal consent by Li Ta-Tsien (Editor-in-Chief of Chinese Ann. Math.). Translated into English by Hajime Urakawa.

Constant maps and harmonic maps are trivial examples of 2-harmonic maps. The main results of §3 are to give nontrivial examples of 2-harmonic maps. We consider Riemannian isometric immersions. For the isometric immersions with parallel mean curvature tensor field, we give the decomposition formula (Lemma 4) of the Laplacian of the tension field $\tau(f)$ with its proof: for the hypersurfaces M with non-zero parallel mean curvature tensor field in the unit sphere S^{m+1}, a necessary and sufficient condition for such isometric immersions to be 2-harmonic is that the square of the length of the second fundamental form $B(f)$ satisfies $\|B(f)\|^2 = m$. Using this, special Clifford tori in the unit sphere whose Gauss maps are studied by Y.L. Xin and Q. Chen [2], give non-trivial 2-harmonic maps which are isometric immersions in the unit sphere.

In §4, using formulas in §2, we derive the second variation formula of 2-harmonic maps (Theorem 3) and give the definition of stability of 2-harmonic maps (the second variation is nonnegative) and give a proof of the following (Theorem 4): if M is compact, and N has positive constant sectional curvature, there are no nontrivial 2-harmonic maps from M into N satisfying the conservation law. Last, when $N = \mathbb{C}P^n$ we establish nonexistence results of stable 2-harmonic maps (Lemma 8, Theorem 5, etc.). Furthermore, we give a nonexistence theorem establishing sufficient conditions that stable 2-harmonic maps be harmonic.

Acknowledgements. We would like to express our gratitude to Professors Su Bu-Chin and Hu He-Shen who introduced and helped to accomplish this paper. We also would like to express our thanks to Professors Shen Chun-Li, Xin Yuan-Long and Pan Yang-Lian who helped us during the period of our study.

1 Notation, and fundamental notions

We prepare the main materials using vector bundle valued differential forms and Riemannian metrics on bundles which are in [1, 3].

Assume that (M, g) is a m-dimensional Riemannian manifold, (N, h) a n-dimensional one, and $f : M \to N$ a C^∞ map. Given points $p \in M$ and $f(p) \in N$, under (x^i), (y^α) local coordinates around them, f can be expressed as

$$y^\alpha = f^\alpha(x^i), \tag{1}$$

where the indices we use run as follows

$$i, j, k, \cdots = 1, \cdots, m;\ \alpha, \beta, \gamma, \cdots = 1, \cdots, n.$$

We use the following definition: the differential df of f can be regarded as the induced bundle $f^{-1}TN$-valued 1-form

$$df(X) = f_*X, \quad \forall X \in \Gamma(TM). \tag{2}$$

We denote by f^*h the first fundamental form of f, which is a section of the symmetric bilinear tensor bundle $\bigodot^2 T^*M$; the second fundamental form $B(f)$ of f is the covariant derivative $\widetilde{\nabla} df$ of the 1-form df, which is a section of $\bigodot^2 T^*M \otimes f^{-1}TN$:

$$\begin{aligned} &\forall X, Y \in \Gamma(TM): \\ B(f)(X,Y) &= (\widetilde{\nabla} df)(X,Y) = (\widetilde{\nabla}_X df)(Y) = \\ &= \overline{\nabla}_X df(Y) - df(\nabla_X Y) = \\ &= \nabla'_{df(X)} df(Y) - df(\nabla_X Y). \end{aligned} \tag{3}$$

Here ∇, ∇', $\overline{\nabla}$, $\widetilde{\nabla}$ are the Riemannian connections on the bundles TM, TN, $f^{-1}TN$ and $T^*M \otimes f^{-1}TN$, respectively. From $\widetilde{\nabla} df$, by using a local orthonormal frame field $\{e_i\}$ on M, one obtains the tension field $\tau(f)$ of f

$$\tau(f) = (\widetilde{\nabla} df)(e_i\,,\, e_i) = (\widetilde{\nabla}_{e_i} df)(e_i). \tag{4}$$

In the following, we use the above notations without comments, and we assume the reader is familiar with the above notation.

We say f is a harmonic map if $\tau(f) = 0$. If M is compact, we consider critical maps of the energy functional

$$E(f) = \int_M \|df\|^2 * 1, \tag{5}$$

where $\frac{1}{2}\|df\|^2 = \frac{1}{2}\langle df(e_i), df(e_i)\rangle_N = e(f)$ which is called the energy density of f, and the inner product $\langle\,,\,\rangle_N$ is a Riemannian metric h, and we omit the subscript N if there is no confusion. When f is an isometric immersion, $\frac{1}{m}\tau(f)$ is the mean curvature normal vector field and harmonic maps are minimal immersions.

The curvature tensor field $\widetilde{R}(\,,\,)$ of the Riemannian metric on the bundle $T^*M \otimes f^{-1}TN$ is defined as follows

$$\begin{aligned} &\forall X, Y \in \Gamma(TM): \\ &\widetilde{R}(X,Y) = -\widetilde{\nabla}_X \widetilde{\nabla}_Y + \widetilde{\nabla}_Y \widetilde{\nabla}_X + \widetilde{\nabla}_{[X,Y]}. \end{aligned} \tag{6}$$

Furthermore, for any $Z \in \Gamma(TM)$, we define

$$(\widetilde{R}(X,Y)df)(Z) = R^{f^{-1}TN}(X,Y)df(Z) - df(R^M(X,Y)Z) =$$

$$= R^N(df(X), df(Y))df(Z) - df(R^M(X,Y)Z), \tag{7}$$

where R^M, R^N, and $R^{f^{-1}TN}$ are the Riemannian curvature tensor fields on TM, TN, $f^{-1}TN$, respectively.

For 1-forms df the Weitzenböck formula is given by

$$\Delta df = \widetilde{\nabla}^* \widetilde{\nabla} df + S, \tag{8}$$

where $\Delta = dd^* + d^*d$ is the Hodge-Laplace operator, $-\widetilde{\nabla}^*\widetilde{\nabla} = \widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} - \widetilde{\nabla}_{\nabla_{e_k}e_k}$ is the rough Laplacian, and the operator S is defined as follows

$$\forall X \in \Gamma(TM):$$

$$S(X) = -(\widetilde{R}(e_k, X)df)(e_k), \tag{9}$$

where $\{e_k\}$ is a locally defined orthonormal frame field on M.

A section of $\bigodot^2 T^*M$ defined by $S_f = e(f)g - f^*h$ is called the stress-energy tensor field, and f is said to satisfy the conservation law if $\mathrm{div}S_f = 0$. As in [1], $\forall X \in \Gamma(TM)$, it holds that

$$(\mathrm{div}S_f)(X) = -\langle \tau(f), df(X)\rangle. \tag{10}$$

Maps satisfying the conservation law are said to be *relatively harmonic* ([6]).

2 The first variation formula of 2-harmonic maps

Assume that $f : M \to N$ is a C^∞ map, M is a compact Riemannian manifold, and N is an arbitrary Riemannian manifold. As in [1], a 2-harmonic map is a critical map of the functional

$$E_2(f) = \int_M \|(d + d^*)^2 f\|^2 * 1. \tag{11}$$

Here, d and d^* are the exterior differentiation and the codifferentiation on vector bundle, and $*1$ is the volume form on M.

In order to derive the analytic condition of the 2-harmonic maps, we have to calculate the first variation of $E_2(f)$ defined by (11). To start with let

$$f_t : M \to N, \quad t \in I_\epsilon = (-\epsilon, \epsilon), \;\; \epsilon > 0, \tag{12}$$

be a smooth 1-parameter variation of f which yields a vector field $V \in \Gamma(f^{-1}TN)$ along f in N by

$$f_0 = f, \quad \left.\frac{\partial f_t}{\partial t}\right|_{t=0} = V. \tag{13}$$

Variation $\{f_t\}$ yields a C^∞ map

$$\begin{aligned} F:\, M \times I_\epsilon \to N, \\ F(p,t) = f_t(p), \quad \forall p \in M,\, t \in I_\epsilon. \end{aligned} \tag{14}$$

If we take the local coordinates around $p \in M$, $f_t(p) \in N$, respectively, we have

$$y^\alpha = F^\alpha(x^i, t) = f_t^\alpha(x^i). \tag{15}$$

Taking the usual Euclidean metric on I_ϵ, with respect to the product Riemannian metric on $M \times I_\epsilon$, we denote by $\boldsymbol{\nabla}$, $\overline{\boldsymbol{\nabla}}$, $\widetilde{\boldsymbol{\nabla}}$, the induced Riemann connections on $T(M \times I_\epsilon)$, $F^{-1}TN$, $T^*(M \times I_\epsilon) \otimes F^{-1}TN$, respectively. If $\{e_i\}$ is an orthonormal frame field defined on a neighborhood U of p, $\left\{e_i, \frac{\partial}{\partial t}\right\}$ is also an orthonormal frame field on a coordinate neighborhood $U \times I_\epsilon$ in $M \times I_\epsilon$, and it holds that

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0,\, \boldsymbol{\nabla}_{e_i} e_j = \nabla_{e_i} e_j,\, \boldsymbol{\nabla}_{\frac{\partial}{\partial t}} e_i = \boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t} = 0. \tag{16}$$

It also holds that

$$\frac{\partial f_t}{\partial t} = \frac{\partial F^\alpha}{\partial t} \frac{\partial}{\partial y^\alpha} = dF\left(\frac{\partial}{\partial t}\right),\, df_t(e_i) = dF(e_i), \tag{17}$$

and

$$\begin{aligned} (\widetilde{\nabla}_{e_i} df_t)(e_j) &= \nabla'_{df_t(e_i)} df_t(e_j) - df_t(\nabla_{e_i} e_j) = (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_j) \\ (\widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_i} df_t)(e_j) &= \nabla'_{df_t(e_k)}((\widetilde{\nabla}_{e_i} df_t)(e_j)) - (\widetilde{\nabla}_{e_i} df_t)(\nabla_{e_k} e_j) \\ &= (\widetilde{\boldsymbol{\nabla}}_{e_k} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_j) \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{18}$$

etc. Here, we used the abbreviated symbol $\widetilde{\nabla}$ on $T^*M \otimes f_t^{-1}TN$ in which we omitted t.

In the following, we need two lemmas to calculate the first variation

$$\frac{d}{dt} E_2(f_t)|_{t=0}$$

of $E_2(f)$.

1 Lemma. *Under the above notation, for any C^∞ variation $\{f_t\}$ of f, it holds that*

$$\begin{aligned} &\frac{d}{dt} E_2(f_t) = \\ &2\int_M \left\langle (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j) \right\rangle * 1 \\ &+ 2\int_M \left\langle R^N\left(dF(e_i), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_i), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j) \right\rangle * 1. \end{aligned} \tag{19}$$

Proof. By using d, d^*, and definition of $\tau(f)$, (11) can be written as

$$\begin{aligned} E_2(f) &= \int_M \|d^* df\|^2 * 1 = \int_M \|\tau(f)\|^2 * 1 \\ &= \int_M \langle (\widetilde{\nabla}_{e_i} df)(e_i), \widetilde{\nabla}_{e_i} df)(e_i)\rangle * 1. \end{aligned} \tag{20}$$

By noting (18), for variation f_t of f, it holds that

$$\begin{aligned} \frac{d}{dt} E_2(f_t) &= \frac{d}{dt} \int_M \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j) \ \rangle * 1 \\ &= 2 \int_M \langle \widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} ((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle * 1. \end{aligned} \tag{21}$$

By (16), and using the curvature tensor on $T^*(M \times I_\epsilon) \otimes F^{-1}TN$,

$$\begin{aligned} \left(\widetilde{R}\left(X, \frac{\partial}{\partial t}\right) dF\right)(Y) &= R^N \left(dF(X), dF\left(\frac{\partial}{\partial t}\right)\right) dF(Y) \\ &\quad - dF\left(R^{M\times I_\epsilon}\left(X, \frac{\partial}{\partial t}\right) Y\right) \\ &= R^N \left(dF(X), dF\left(\frac{\partial}{\partial t}\right)\right) dF(Y), \end{aligned} \tag{22}$$

for all $X, Y \in \Gamma(TM)$. In (21), interchanging the order of differentiations in $\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} ((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)$, we have

$$\begin{aligned} \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} ((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)) &= (\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i) \\ &= \left(\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} dF - \widetilde{\boldsymbol{\nabla}}_{[e_i, \frac{\partial}{\partial t}]} dF + \widetilde{R}\left(e_i, \frac{\partial}{\partial t}\right) dF\right)(e_i) \\ &= \widetilde{\boldsymbol{\nabla}}_{e_i} ((\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} dF)(e_i)) - (\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} dF)(\nabla_{e_i} e_i) \\ &\quad + R^N \left(dF(e_i), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_i) \\ &= (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} dF)\left(\frac{\partial}{\partial t}\right) \\ &\quad + R^N \left(dF(e_i), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_i). \end{aligned} \tag{23}$$

In the last of the above, we used the symmetry of the second fundamental form.

By substituting (23)[1] into (21) we obtain (19). $\boxed{QED}$

2 Lemma.

$$\int_M \left\langle (\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right\rangle * 1$$
$$= \int_M \left\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_k}\overline{\boldsymbol{\nabla}}_{e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)) - \overline{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\right\rangle * 1. \quad (24)$$

Proof. For each $t \in I_\epsilon$, let us define a C^∞ vector field on M by

$$X = \left\langle (\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right\rangle e_i, \quad (25)$$

which is well defined because of the independence on a choice of $\{e_i\}$. The divergence of X is given by

$$\mathrm{div}X = \langle \nabla_{e_k}X, e_k\rangle_M = \nabla_{e_i}\left\langle (\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right\rangle$$
$$+ \left\langle (\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right\rangle \langle \nabla_{e_k}e_i, e_k\rangle_M. \quad (26)$$

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[1] Translator's comments: to get the last equation of (23), we have to see that

$$\begin{aligned}(\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right) &= \overline{\boldsymbol{\nabla}}_{e_i}\left(dF\left(\frac{\partial}{\partial t}\right)\right) - dF\left(\boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}\right)\\ &= \overline{\boldsymbol{\nabla}}_{dF(e_i)}dF\left(\frac{\partial}{\partial t}\right)\\ &= \overline{\boldsymbol{\nabla}}_{dF(\frac{\partial}{\partial t})}dF(e_i) - \overline{\boldsymbol{\nabla}}_{[dF(e_i),dF(\frac{\partial}{\partial t})]}\\ &= (\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF)(e_i),\end{aligned}$$

and by a similar way,

$$(\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF)(\nabla_{e_i}e_i) = (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)\left(\frac{\partial}{\partial t}\right).$$

Thus,

$$\widetilde{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF)(e_i))) - (\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF)(\nabla_{e_i}e_i)$$

coincides with

$$\begin{aligned}&\overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right)\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)\left(\frac{\partial}{\partial t}\right)\\ &\quad= (\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)\left(\frac{\partial}{\partial t}\right)\\ &\quad= (\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)\left(\frac{\partial}{\partial t}\right),\end{aligned}$$

which implies (23).

Noticing (16) and

$$\langle \nabla_{e_k} e_i, e_k \rangle_M + \langle e_i, \nabla_{e_k} e_k \rangle_M = 0, \tag{27}$$

we have (28)[2]:

$$\begin{aligned} \mathrm{div}(X) &= \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \\ &\quad + \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j))\rangle \\ &\quad - \langle \left(\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} dF\right)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle. \end{aligned} \tag{28}$$

Furthermore, let us define a C^∞ vector field Y on M by

$$Y = \left\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)) \right\rangle e_i, \tag{29}$$

which is also well defined. Then, by a similar way, we have

$$\begin{aligned} \mathrm{div} Y &= \langle \nabla_{e_k} Y, e_k \rangle_M \\ &= \left\langle (\widetilde{\boldsymbol{\nabla}}_{e_k} dF)\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)) \right\rangle \end{aligned}$$

[2]Translator's comments: the first term of (26) coincides with

$$\begin{aligned} &\langle \overline{\boldsymbol{\nabla}}_{e_i} (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \\ &\qquad + \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), \overline{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j))\rangle \\ &= \langle \widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF(\frac{\partial}{\partial t}) + (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \\ &\quad + \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), \overline{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}} dF)(e_j))\rangle \\ &= \langle \widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF(\frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \\ &\quad + \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), \overline{\boldsymbol{\nabla}}_{e_i}((\widetilde{\boldsymbol{\nabla}} dF)(e_j))\rangle, \end{aligned}$$

and the second term of (26) coincides with

$$\begin{aligned} &\langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \langle \nabla_{e_k} e_i, e_k \rangle_M \\ &\qquad = -\langle (\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle \langle e_i, \nabla_{e_k} e_k \rangle_M \\ &\qquad = -\langle (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} dF)(\frac{\partial}{\partial t}), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\rangle. \end{aligned}$$

Thus, we have (28).

$$+\left\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_k}\overline{\boldsymbol{\nabla}}_{e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\right\rangle$$
$$-\left\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\right\rangle. \tag{30}$$

By the Green's theorem, we have

$$\int_M \operatorname{div}(X-Y)*1=0, \tag{31}$$

and together with (28) and (30), we have (24). $\boxed{QED}$

3 Theorem. *Assume that $f : M \to N$ is a C^∞ map from a compact Riemannian manifold M into an arbitrary Riemannian manifold N, $\{f_t\}$ is an arbitrary C^∞ variation generating V. Then,*

$$\left.\frac{d}{dt}E_2(f_t)\right|_{t=0} =$$
$$=2\int_M \langle V, -\overline{\nabla}^*\overline{\nabla}\tau(f)+R^N(df(e_i),\tau(f))df(e_i)\rangle*1. \tag{32}$$

Proof. Substituting (24) into (19), we have

$$\frac{d}{dt}E_2(f_t)=2\int_M\left\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_k}\overline{\boldsymbol{\nabla}}_{e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)) -\overline{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\right\rangle*1$$
$$+2\int_M\langle R^N\left(dF(e_i), dF\left(\frac{\partial}{\partial t}\right)\right)dF(e_i), (\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\rangle*1, \tag{33}$$

where putting $t=0$, noticing (13), (17), (18), and the symmetry of the curvature tensor, we have (32). Here, we used the explicit formula of the rough Laplacian on $f^{-1}TN$, that is $-\overline{\nabla}^*\overline{\nabla}=\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}-\overline{\nabla}_{\nabla_{e_k}e_k}$. $\boxed{QED}$

4 Remark. In the above arguments, we assumed M is a compact Riemannian manifold without boundary. For a general Riemannian manifold M, let $\mathcal{D}\subset M$ be an arbitrarily bounded domain with smooth boundary, and take a variation $\{f_t\}$ of f satisfying that

$$\left.\frac{\partial f_t}{\partial t}\right|_{\partial\mathcal{D}}=0, \quad \left.\left(\overline{\nabla}_{e_i}\frac{\partial f_t}{\partial t}\right)\right|_{\partial\mathcal{D}}=0,$$

then, in Lemma 2, we obtain (24) by applying the Green's divergence theorem to X and Y. Then, we have the first variational formula on $\mathcal{D}$ as

$$\left.\frac{d}{dt}E_2(f_t,\mathcal{D})\right|_{t=0}=2\int_{\mathcal{D}}\langle V, -\overline{\nabla}^*\overline{\nabla}\tau(f)+R^N(df(e_i),\tau(f))df(e_i)\rangle*1,$$

where $E_2(f_t,\mathcal{D})$ is the corresponding functional relative to $\mathcal{D}$.

5 Definition. For a C^∞ map $f : M \to N$ between two Riemannian manifolds, let us define the *2-tension field* $\tau_2(f)$ of f by

$$\tau_2(f) = -\overline{\nabla}^*\overline{\nabla}\tau(f) + R^N(df(e_i), \tau(f))df(e_i). \tag{34}$$

f is said to be a *2-harmonic map* is if $\tau_2(f) = 0$.

The C^∞ function

$$e_2(f) = \frac{1}{2}\|(d+d^*)^2 f\|^2 = \frac{1}{2}\|\tau(f)\|^2, \tag{35}$$

is called the *2-energy density*, and

$$\frac{1}{2}E_2(f) = \int_M e_2(f) * 1 < +\infty,$$

is the *2-energy* of f. If M is compact, by the first variational formula, a 2-harmonic map f is a critical point of the 2-energy.

3 Examples of 2-harmonic maps

By Definition 1, we have immediately

6 Proposition. (1) *Any harmonic map is 2-harmonic.*

(2) *Any doubly harmonic function $f : M \to \mathbb{R}$ on a Riemannian manifold M is also 2-harmonic.*

7 Proposition. *Assume that M is compact and N has non positive curvature, i.e.* $\mathrm{Riem}^N \leq 0$. *Then every 2-harmonic map $f : M \to N$ is harmonic.*

Proof. Computing the Laplacian of the 2-energy density $e_2(f)$ we have

$$\begin{aligned}\Delta e_2(f) &= \frac{1}{2}\Delta\|\tau(f)\|^2 \\ &= \langle \overline{\nabla}_{e_k}\tau(f), \overline{\nabla}_{e_k}\tau(f)\rangle + \langle -\overline{\nabla}^*\overline{\nabla}\tau(f), \tau(f)\rangle.\end{aligned} \tag{36}$$

Taking

$$\tau_2(f) = -\overline{\nabla}^*\overline{\nabla}\tau(f) + R^N(df(e_i), \tau(f))df(e_i) = 0,$$

and noticing $\mathrm{Riem}^N \leq 0$, we have

$$\begin{aligned}\Delta e_2(f) &= \langle \nabla_{e_k}\tau(f), \overline{\nabla}_{e_k}\tau(f)\rangle - R^N(df(e_i), \tau(f))df(e_i), \tau(f)\rangle \\ &\geq 0.\end{aligned} \tag{37}$$

By the Green's theorem $\int_M \Delta e_2(f) v_g = 0$, and (37), we have $\Delta e_2(f) = 0$, so that $e_2(f) = \frac{1}{2}\|\tau(f)\|^2$ is constant. Again, by (37), we have

$$\overline{\nabla}_{e_k}\tau(f) = 0, \quad \forall\, k = 1, \cdots, m.$$

Therefore, by [1], we have[3] $\tau(f)=0$. $\boxed{QED}$

8 Remark. As we know nonexistence of compact minimal submanifolds in the Euclidean space, Proposition 2 shows nonexistence of 2-harmonic isometric immersions from compact Riemannian manifolds.

By Proposition 1 harmonic maps are trivial examples of 2-harmonic ones, and in Proposition 2 in the case that M is compact and the sectional curvature of N does not have nonpositive curvature, one may ask examples of nontrivial 2-harmonic maps. To do it, the following lemmas complete this.

9 Lemma. *Assume that $f : M \to N$ is a Riemannian isometric immersion whose mean curvature vector field is parallel. Then, for a locally defined orthonormal frame field $\{e_i\}$, we have*

$$\begin{aligned} -\overline{\nabla}^*\overline{\nabla}\tau(f) &= \langle -\overline{\nabla}^*\overline{\nabla}\tau(f), df(e_i)\rangle df(e_i) \\ &\quad + \langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle\, (\widetilde{\nabla}_{e_i}df)(e_j). \end{aligned} \tag{38}$$

Proof. Since f is an isometric immersion, $df(e_i)$ span the tangent space of $f(M)\subset N$. Since[4] the mean curvature tensor is parallel, for all $i=1,\cdots,m$, $\overline{\nabla}_{e_i}\tau(f)\in\Gamma(f_*TM)$. Thus

$$\overline{\nabla}_{e_i}\tau(f) = \langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle df(e_j). \tag{39}$$

[3]Translator's comments: since $\Delta e_2(f)=0$, both terms of (37) are non negative, we have $\langle \overline{\nabla}_{e_k}\tau(f), \overline{\nabla}_{e_k}\tau(f)\rangle = 0$, i.e., $\overline{\nabla}_{e_k}\tau(f)=0$ for all $k=1,\cdots,m$. We can define a global vector field $X_f = \langle df(e_i), \tau(f)\rangle e_i \in \mathfrak{X}(M)$, whose divergence is given as

$$\mathrm{div}(X_f) = \langle \tau(f),\tau(f)\rangle + \langle df(e_i), \overline{\nabla}_{e_i}\tau(f)\rangle = \langle \tau(f),\tau(f)\rangle.$$

Integrating this over M, we have

$$0 = \int_M \mathrm{div}(X_f) v_g = \int_M \langle \tau(f),\tau(f)\rangle v_g,$$

which implies $\tau(f)=0$.

[4]Translator's comments: for all $\xi\in\Gamma(T^\perp N)$,

$$\overline{\nabla}_X\xi = \nabla'_{f_*X}\xi = \nabla^T_{f_*X}\xi + \nabla^\perp_{f_*X}\xi \in TM + T^\perp M,$$

respectively. The condition that the mean curvature tensor is parallel means that

$$\nabla^\perp_{f_*X}\tau(f) = 0, \quad \forall X\in\mathfrak{X}(M),$$

which is equivalent to the condition that

$$\overline{\nabla}_X\tau(f) = \nabla^T_{f_*X}\tau(f) \in \Gamma(f_*TM).$$

Calculating this, we have

$$
\begin{aligned}
-\overline{\nabla}^*\overline{\nabla}\tau(f) &= \langle -\overline{\nabla}^*\overline{\nabla}\tau(f), df(e_j)\rangle df(e_j) \\
&\quad + \langle \overline{\nabla}_{e_i}\tau(f), \overline{\nabla}_{e_i}df(e_j)\rangle df(e_j) \\
&\quad + \langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle \overline{\nabla}_{e_i}df(e_j).
\end{aligned} \tag{40}
$$

Here, if we denote $\nabla_{e_i}e_j = \Gamma^k_{ij}e_k$, we have $\Gamma^j_{ki} + \Gamma^i_{kj} = e_k\langle e_i, e_j\rangle = 0$. Since

$$
(\widetilde{\nabla}_{e_i}df)(e_j) = \overline{\nabla}_{e_i}(df(e_j)) - df(\nabla_{e_i}e_j) \in T^\perp M \subset TN,
$$

and $\overline{\nabla}_{e_i}\tau(f) \in f_*(TM)$, the second term of (40) is

$$
\begin{aligned}
\langle \overline{\nabla}_{e_i}\tau(f), \overline{\nabla}_{e_i}df(e_j)\rangle df(e_j) &= \langle \overline{\nabla}_{e_i}\tau(f), (\widetilde{\nabla}_{e_i}df)(e_j) + df(\nabla_{e_i}e_j)\rangle df(e_j) \\
&= \langle \overline{\nabla}_{e_i}\tau(f), df(\nabla_{e_i}e_j)\rangle df(e_j) \\
&= \langle \overline{\nabla}_{e_i}\tau(f), df(e_k)\rangle\, df(\Gamma^k_{ij}e_j) \\
&= \langle \overline{\nabla}_{e_i}\tau(f), df(e_k)\rangle\, df(-\Gamma^j_{ik}e_j) \\
&= -\langle \overline{\nabla}_{e_i}\tau(f), df(e_k)\rangle\, df(\nabla_{e_i}e_k).
\end{aligned} \tag{41}
$$

Substituting (41) into (40), we have (38)[5]. $\boxed{QED}$

10 Lemma. *For an isometric immersion $f : M \to N$ with parallel mean curvature vector field, the Laplacian of $\tau(f)$ is decomposed into:*

$$
\begin{aligned}
-\overline{\nabla}^*\overline{\nabla}\tau(f) &= \langle \tau(f), R^N(df(e_k), df(e_j))df(e_k)\rangle df(e_j) \\
&\quad - \langle \tau(f), (\widetilde{\nabla}_{e_i}df)(e_j)\rangle (\widetilde{\nabla}_{e_i}df)(e_j).
\end{aligned} \tag{42}
$$

Proof. Calculate the right hand side of (38). By differentiating by

$$
e_i\langle \tau(f), df(e_j)\rangle = 0\,,
$$

we have

$$
\langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle + \langle \tau(f), \overline{\nabla}_{e_i}df(e_j)\rangle = e_i\langle \tau(f), df(e_j)\rangle = 0. \tag{43}
$$

[5]Translator's comments: for (38), we only have to see

$$
\begin{aligned}
-\overline{\nabla}^*\overline{\nabla}\tau(f) &= \langle -\overline{\nabla}^*\overline{\nabla}\tau(f), df(e_i)\rangle df(e_i) \\
&\quad + \langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle \{\overline{\nabla}_{e_i}df(e_j) - df(\nabla_{e_i}e_j)\} \\
&= \langle -\overline{\nabla}^*\overline{\nabla}\tau(f), df(e_i)\rangle df(e_i) \\
&\quad + \langle \overline{\nabla}_{e_i}\tau(f), df(e_j)\rangle (\widetilde{\nabla}_{e_i}df)(e_j)
\end{aligned}
$$

by (3), which is (38).

Then, we have

$$\begin{aligned}\langle\overline{\nabla}_{e_i}\tau(f),df(e_j)\rangle &= -\langle\tau(f),\overline{\nabla}_{e_i}df(e_j)\rangle\\ &= -\langle\tau(f),\overline{\nabla}_{e_i}df(e_j)-df(\nabla_{e_i}e_j)\rangle\\ &= -\langle\tau(f),(\widetilde{\nabla}_{e_i}df)(e_j)\rangle. \end{aligned}\tag{44}$$

For the first term of the RHS of (38), by differentiating (43) by e_i, we have

$$\begin{aligned}\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(f),&df(e_j)\rangle+2\langle\overline{\nabla}_{e_i}\tau(f),\overline{\nabla}_{e_i}df(e_j)\rangle\\ &+\langle\tau(f),\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}df(e_j)\rangle=0.\end{aligned}\tag{45}$$

We also have

$$\langle\overline{\nabla}_{\nabla_{e_i}e_i}\tau(f),df(e_j)\rangle+\langle\tau(f),\overline{\nabla}_{\nabla_{e_i}e_i}df(e_j)\rangle=\nabla_{e_i}e_i\langle\tau(f),df(e_j)\rangle=0.\tag{46}$$

Together with (45) and (46), we have

$$\begin{aligned}\langle-\overline{\nabla}^*\overline{\nabla}\tau(f),df(e_j)\rangle&+2\langle\overline{\nabla}_{e_i}\tau(f),\overline{\nabla}_{e_i}df(e_j)\rangle\\ &+\langle\tau(f),-\overline{\nabla}^*\overline{\nabla}df(e_j)\rangle=0.\end{aligned}\tag{47}$$

For the second term of (47), by making use of the fact that $\overline{\nabla}_{e_i}\tau(f)\in\Gamma(f_*TM)$ from the assumption that the mean curvature tensor is parallel, and (44), we have

$$\begin{aligned}\langle\overline{\nabla}_{e_i}\tau(f),\overline{\nabla}_{e_i}df(e_j)\rangle &= \langle\overline{\nabla}_{e_i}\tau(f),(\widetilde{\nabla}_{e_i}df)(e_j)+df(\nabla_{e_i}e_j)\rangle\\ &= \langle\overline{\nabla}_{e_i}\tau(f),df(\nabla_{e_i}e_j)\rangle\\ &= -\langle\tau(f),(\widetilde{\nabla}_{e_i}df)(\nabla_{e_i}e_j)\rangle.\end{aligned}\tag{48}$$

For the third term of (47), we have

$$\begin{aligned}\langle\tau(f),-\overline{\nabla}^*\overline{\nabla}df(e_j)\rangle &= \langle\tau(f),\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}df(e_j)-\overline{\nabla}_{e_k}df(e_j)\rangle\\ &= \langle\tau(f),\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_k}df)(e_j)+df(\nabla_{e_k}e_j))\\ &\qquad -(\widetilde{\nabla}_{\nabla_{e_k}e_k}df)(e_j)-df(\nabla_{\nabla_{e_k}e_k}e_j)\rangle\\ &= \langle\tau(f),(\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k}df)(e_j)\\ &\qquad +2(\widetilde{\nabla}_{e_k}df)(\nabla_{e_k}e_j)-(\widetilde{\nabla}_{\nabla_{e_k}e_k}e_j)\rangle\\ &= \langle\tau(f),(-\widetilde{\nabla}^*\widetilde{\nabla}df)(e_j)\rangle\\ &\qquad +2\langle\tau(f),(\widetilde{\nabla}_{e_k}df)(\nabla_{e_k}e_j)\rangle\\ &= \langle\tau(f),-\Delta df(e_j)+S(e_j)\rangle\\ &\qquad +2\langle\tau(f),(\widetilde{\nabla}_{e_k}df)(\nabla_{e_k}e_j)\rangle\end{aligned}$$

since $\langle \tau(f), df(X)\rangle = 0$ for all $X \in \mathfrak{X}(M)$ and Weitzenböck formula (8). Here, we have

$$-\Delta df(e_j) = -dd^* df(e_j) = d\tau(f)(e_j) = \overline{\nabla}_{e_j} df,$$

and by (9) and (7),

$$\begin{aligned} S(e_j) &= -\sum_{k=1}^{m} (\widetilde{R}(e_k, e_j) df)(e_k) \\ &= -\sum_{k=1}^{m} \{R^N(df(e_k), df(e_j)) df(e_k) - df(R^M(e_k, e_j)e_k))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \langle \tau(f), \overline{\nabla}^* \overline{\nabla} df(e_j)\rangle &= 2\langle \tau(f), (\widetilde{\nabla}_{e_k} df)(\nabla_{e_k} e_j)\rangle \\ &\quad + \langle \tau(f), \overline{\nabla}_{e_j} \tau(f) - R^N(df(e_k), df(e_j)) df(e_k) + df(R^M(e_k, e_j)e_k)\rangle \\ &= 2\langle \tau(f), (\widetilde{\nabla}_{e_k} df)(\nabla_{e_k} e_j)\rangle \\ &\qquad - \langle \tau(f), R^N(df(e_k), df(e_j)) df(e_k) \end{aligned} \tag{49}$$

since $\overline{\nabla}_{e_j} \tau(f) \in \Gamma(f_* TM)$. Substituting (48) and (49) into (47), we have

$$\langle -\overline{\nabla}^* \overline{\nabla} \tau(f), df(e_j)\rangle = \langle \tau(f), R^N(df(e_k), df(e_j)) df(e_k)\rangle. \tag{50}$$

Finally, substituting (44) and (50) into (38), we have (42). $\boxed{QED}$

Taking for M to be a hypersurface in the unit sphere $N = S^{m+1}$ with $n = m+1$, we have

11 Theorem. *Let $f : M \to S^{m+1}$ be an isometric immersion having parallel mean curvature vector field with non-zero mean curvature. Then, the necessary and sufficient condition for f to be 2-harmonic is $\|B(f)\|^2 = m = \dim M$.*

Proof. Since S^m has constant sectional curvature, the normal component of $R^N(df(e_k), df(e_j)) df(e_k)$ is zero, (42) in Lemma 4 becomes

$$\overline{\nabla}^* \overline{\nabla} \tau(f) = -\langle \tau(f), (\widetilde{\nabla}_{e_i} df)(e_j)\rangle (\widetilde{\nabla}_{e_i} df)(e_j).$$

Noticing $R^N(df(e_k), \tau(f)) df(e_k) = m\tau(f)$, the condition for f to be 2-harmonic becomes

$$-\langle \tau(f), (\widetilde{\nabla}_{e_i} df)(e_j)\rangle (\widetilde{\nabla}_{e_i} df)(e_j) + m\tau(f) = 0. \tag{51}$$

Denoting by ξ, the unit normal vector field on $f(M)$, and

$$(\widetilde{\nabla}_{e_i} df)(e_j) = B(f)(e_i, e_j) = H_{ij}\xi$$

in (3), we have $\tau(f) = H_{ii}\xi$ which implies

$$\|\tau(f)\|^2 = H_{ii}H_{jj}, \quad \|B(f)\|^2 = B(f)(e_i, e_j) = H_{ij}H_{ij}.$$

Substituting these into (51), we have

$$(mH_{kk} - H_{kk}H_{ij}H_{ij})\xi = 0,$$

which is equivalent to

$$(m - \|B(f)\|^2)\|\tau(f)\| = 0. \tag{52}$$

Since $\|\tau(f)\| \neq 0$, the condition $\|B(f)\|^2 = m$ is equivalent to 2-harmonicity. $\boxed{QED}$

12 Example. Due to Theorem 2, we can obtain non-trivial examples of 2-harmonic maps. Consider the Clifford torus in the unit sphere S^{m+1}:

$$M_k^m(1) = S^k\left(\sqrt{\frac{1}{2}}\right) \times S^{m-k}\left(\sqrt{\frac{1}{2}}\right),$$

where the integer k satisfies $0 \leq k \leq m$ ([2]). The isometric embeddings $f : M_k^m(1) \to S^{m+1}$ with $k \neq \frac{m}{2}$ are non-trivial 2-harmonic maps. Indeed, f has the parallel second fundamental form, and parallel mean curvature vector field, and by direct computation, we have $\|B(f)\|^2 = k + m - k = m$, $\|\tau(f)\| = |k - (m-k)| = |2k - m| \neq 0$, so by Theorem 2, f is a nontrivial 2-harmonic map.

4 The second variation of 2-harmonic maps

Assume that M is compact, $f : M \to N$ is a 2-harmonic map. We will compute the second variation formula. By using the variation formula in §2 and notation, we continue to calculate (33):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_2(f_t) = \int_M \Big\langle dF\left(\frac{\partial}{\partial t}\right), &\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j}dF)(e_j)) \\ &- \overline{\nabla}_{\nabla_{e_k}e_k}((\widetilde{\nabla}_{e_j}dF)(e_j)) \\ &+ R^N(dF(e_i), (\widetilde{\nabla}_{e_j}dF)(e_j)dF(e_i)\Big\rangle * 1.\end{aligned} \tag{53}$$

Differentiating (53) by t, we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t) = \int_M \Big\langle \overline{\nabla}_{\frac{\partial}{\partial t}} dF\left(\frac{\partial}{\partial t}\right), \overline{\nabla}_{e_k}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ - \overline{\nabla}_{\nabla_{e_k}e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ + R^N(dF(e_i), (\widetilde{\nabla}_{e_j} dF)(e_j) dF(e_i)\Big\rangle * 1 \\ + \int_M \Big\langle dF\left(\frac{\partial}{\partial t}\right), \overline{\nabla}_{\frac{\partial}{\partial t}}\Big[\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ - \overline{\nabla}_{\nabla_{e_k}e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ + R^N(dF(e_i), (\widetilde{\nabla}_{e_j} dF)(e_j) dF(e_i)\Big]\Big\rangle * 1. \qquad (54)\end{aligned}$$

We need two Lemmas to calculate the covariant differentiation with respect to $\frac{\partial}{\partial t}$ the second term of RHS of (54).

13 Lemma.

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) = \overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\Big[(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} dF)\left(\frac{\partial}{\partial t}\right) \\ - (\widetilde{\nabla}_{\nabla_{e_i}e_i} dF)\left(\frac{\partial}{\partial t}\right) \\ + R^N\left(dF(e_j), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_j)\Big] \\ + \overline{\nabla}_{e_k}\Big[R^N\left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right)((\widetilde{\nabla}_{e_j} dF)(e_j))\Big] \\ + R^N\left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right)\overline{\nabla}((\widetilde{\nabla} dF)(e_j)). \qquad (55)\end{aligned}$$

Proof. Let us make use of the curvature formula in $F^{-1}TN$ changing variables:

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_k} = \overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial}{\partial t}} + R^N\left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right). \qquad (56)$$

Using twice this formula, we have

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) = \overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ + R^N\left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right)\overline{\nabla}_{e_k}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ = \overline{\nabla}_{e_k}\Big[\overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial}{\partial t}}((\widetilde{\nabla}_{e_j} dF)(e_j)) \\ + R^N\left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right)((\widetilde{\nabla}_{e_j} dF)(e_j))\Big]\end{aligned}$$

$$+ R^N \left(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)\right) \overline{\boldsymbol{\nabla}}_{e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)).$$

Here, substituting (23) into $\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j))$ in the first term of the RHS, we have (55). $\boxed{QED}$

14 Lemma.

$$\begin{aligned}\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \overline{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k}((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)) = \overline{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} \Bigg[& (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right) \\ & - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} dF)\left(\frac{\partial}{\partial t}\right) \\ & + R^N \left(dF(e_j), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_j) \Bigg] \\ + R^N \left(dF(\nabla_{e_k} e_k), dF\left(\frac{\partial}{\partial t}\right)\right) & ((\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)). \end{aligned} \tag{57}$$

Proof. In a similar way as Lemma 5, since $[\nabla_{e_k} e_k, \frac{\partial}{\partial t}] = 0$, we have

$$\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \overline{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} = \overline{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} + R^N \left(dF(\nabla_{e_k} e_k), dF\left(\frac{\partial}{\partial t}\right)\right).$$

Changing variables, and substituting again (23), we have (57). $\boxed{QED}$

15 Lemma.

$$\begin{aligned} & \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \left[R^N(dF(e_i), (\widetilde{\boldsymbol{\nabla}} dF)(e_j)) dF(e_i) \right] \\ & \quad = (\nabla'_{dF(e_i)} R^N) \left(dF\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\right) dF(e_i) \\ & \quad + (\nabla'_{(\nabla_{e_j} dF)(e_j)} R^N) \left(dF(e_i), dF\left(\frac{\partial}{\partial t}\right)\right) dF(e_i) \\ & \quad + R^N \left((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)\right) dF(e_i) \\ & \quad + R^N(dF(e_i), (\widetilde{\boldsymbol{\nabla}}_{e_j} dF)(e_j)) \left((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)\left(\frac{\partial}{\partial t}\right)\right) \\ & \quad + R^N \Bigg(dF(e_i), (\widetilde{\boldsymbol{\nabla}}_{e_k} \widetilde{\boldsymbol{\nabla}}_{e_k} dF)\left(\frac{\partial}{\partial t}\right) \\ & \qquad - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} dF)\left(\frac{\partial}{\partial t}\right) \\ & \qquad + R^N(dF(e_k), dF\left(\frac{\partial}{\partial t}\right)) dF(e_k) \Bigg) dF(e_i). \end{aligned} \tag{58}$$

Proof. We directly compute the LHS of (58). By definition of $\nabla'_{dF(\frac{\partial}{\partial t})}R$, and then by using the second Bianchi identity, (23) and $\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} dF(e_i) = \overline{\boldsymbol{\nabla}}_{e_i} dF(\frac{\partial}{\partial t})$, we have

$$
\begin{aligned}
&\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}\left[R^N(dF(e_i),(\widetilde{\boldsymbol{\nabla}}dF)(e_j))dF(e_i)\right] \\
&\quad = (\nabla'_{dF(\frac{\partial}{\partial t})}R^N)\left(dF(e_i),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)dF(e_i) \\
&\qquad + R^N\left(\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF(e_i),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)dF(e_i) \\
&\qquad + R^N\left(dF(e_i),\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\right)dF(e_i) \\
&\qquad + R^N\left(dF(e_i),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}dF(e_i) \\
&\quad = (\nabla'_{dF(e_i)}R^N)\left(dF\left(\frac{\partial}{\partial t}\right),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)dF(e_i) \\
&\quad + (\nabla'_{(\nabla_{e_j}dF)(e_j)}R^N)\left(dF(e_i),dF\left(\frac{\partial}{\partial t}\right)\right)dF(e_i) \\
&\quad + R^N\left((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)dF(e_i) \\
&\quad + R^N\left(dF(e_i),(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)\right)\left((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)\left(\frac{\partial}{\partial t}\right)\right) \\
&\quad + R^N\Bigg(dF(e_i),(\widetilde{\boldsymbol{\nabla}}_{e_k}\widetilde{\boldsymbol{\nabla}}_{e_k}dF)\left(\frac{\partial}{\partial t}\right) \\
&\qquad\qquad - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k}dF)\left(\frac{\partial}{\partial t}\right) \\
&\qquad\qquad + R^N(dF(e_k),dF\left(\frac{\partial}{\partial t}\right))dF(e_k)\Bigg)dF(e_i).
\end{aligned}
$$

We have (58). $\boxed{QED}$

16 Theorem. *Let $f : M \to N$ be a 2-harmonic map from a compact Riemannian manifold M into an arbitrary Riemannian manifold N, and $\{f_t\}$ an arbitrary C^∞ variation of f satisfying (12) and (13). Then, the second variation formula of $\frac{1}{2}E_2(f_t)$ is given as follows.*

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{dt^2}E_2(f_t)\Big|_{t=0} &= \int_M \langle -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i),V)df(e_i), \\
&\qquad\qquad -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i),V)df(e_i)\rangle * 1 \\
&\quad + \int_M \langle V, (\nabla'_{df(e_i)}R^N)(df(e_i),\tau(f))V
\end{aligned}
$$

$$\begin{aligned}
&+ (\nabla'_{\tau(f)} R^N)(df(e_i), V)df(e_i) \\
&+ R^N(\tau(f), V)\tau(f) \\
&+ 2R^N(df(e_k), V)\overline{\nabla}_{e_k}\tau(f) \\
&+ 2R^N(df(e_i), \tau(f))\overline{\nabla}_{e_i}V\rangle * 1.
\end{aligned} \tag{59}$$

Proof. Putting $t = 0$ in (54), the first term of RHS vanishes since f is 2-harmonic. It suffices to substitute (55), (56) and (55) in Lemmas 5, 6, and 7 into the second term. Then, we have

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = \int_M \langle V, &-\overline{\nabla}^*\overline{\nabla}(-\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i), V)df(e_i)) \\
&+ \overline{\nabla}_{e_k}(R^N(df(e_k), V)\tau(f)) \\
&+ R^N(df(e_k), V)\overline{\nabla}_{e_k}\tau(f) \\
&- R^N(df(\nabla_{e_k}e_k), V)\tau(f) \\
&+ (\nabla'_{df(e_i)} R^N)(V, \tau(f))df(e_i) \\
&+ (\nabla'_{\tau(f)} R^N)(df(e_i), V)df(e_i) \\
&+ R^N(\overline{\nabla}_{e_i}V, \tau(f))df(e_i) \\
&+ R^N(df(e_i), \tau(f))\overline{\nabla}_{e_i}V \\
&+ R^N(df(e_i), -\overline{\nabla}^*\overline{\nabla}V \\
&\qquad + R^N(df(e_j), V)df(e_j))df(e_i)\rangle * 1.
\end{aligned} \tag{60}$$

In the first term of (60), we have by Green's theorem,

$$\begin{aligned}
&\int_M \langle V, -\overline{\nabla}^*\overline{\nabla}(-\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i), V)df(e_i)\rangle * 1 \\
&= \int_M \langle -\overline{\nabla}^*\overline{\nabla}V, -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i), V)df(e_i)\rangle * 1.
\end{aligned} \tag{61}$$

For the last term of the RHS of (60), by the symmetric property of the curvature

$$\int_M \langle V, R^N(df(e_i), W)df(e_i)\rangle * 1 = \int_M \langle W, R^N(df(e_i), V)df(e_i)\rangle * 1,$$

we have

$$\begin{aligned}
&\int_M \langle V, R^N(df(e_i), -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_j), V)df(e_j)df(e_i)\rangle * 1 \\
&\qquad = \int_M \langle R^N(df(e_i), V)df(e_i), \\
&\qquad\qquad - \overline{\nabla}^*\overline{\nabla}V + R^N(df(e_j), V)df(e_j)\rangle * 1.
\end{aligned} \tag{62}$$

For the second term of the RHS of (60), we have

$$\begin{aligned}\overline{\nabla}_{e_k}(R^N(df(e_k),V)\tau(f)) = {} & (\nabla'_{df(e_k)}R^N)(df(e_k),V)\tau(f)\\ & + R^N(\overline{\nabla}_{e_k}df(e_k),V)\tau(f)\\ & + R^N(df(e_k),\overline{\nabla}_{e_k}V)\tau(f)\\ & + R^N(df(e_k),V)\overline{\nabla}_{e_k}\tau(f). \end{aligned} \tag{63}$$

Substituting (61), (62) and (63) into (60), we have

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{dt^2}E_2(f_t)\Big|_{t=0} = {} & \int_M \langle -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i),V)df(e_i),\\ & \qquad -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i),V)df(e_i)\rangle * 1\\ & + \int_M \langle V, (\nabla'_{df(e_i)}R^N)(df(e_i),\tau(f))V\\ & \qquad + R^N(\tau(f),V)\tau(f) + R^N(df(e_k),\overline{\nabla}_{e_k}V)\tau(f)\\ & \qquad + 2R^N(df(e_k),V)\overline{\nabla}_{e_k}\tau(f)\\ & \qquad + (\nabla'_{df(e_i)}R^N)(V,\tau(f))df(e_i)\\ & \qquad + (\nabla'_{\tau(f)}R^N)(df(e_i),V)df(e_i)\\ & \qquad + R^N(\overline{\nabla}_{e_i}V,\tau(f))df(e_i)\\ & \qquad + R^N(df(e_i),\tau(f))\overline{\nabla}_{e_i}V\rangle * 1. \end{aligned} \tag{64}$$

By the first Bianchi identity, we have

$$\begin{aligned}& R^N(df(e_k),\overline{\nabla}_{e_k}V)\tau(f) + R^N(\overline{\nabla}_{e_i}V,\tau(f)df(e_i)\\ & \qquad = R^N(df(e_i),\tau(f))\overline{\nabla}_{e_i}V,\\ & (\nabla'_{df(e_k)}R^N)(df(e_k),V)\tau(f) + (\nabla'_{df(e_k)}R^N)(V,\tau(f))df(e_k)\\ & \qquad = (\nabla'_{df(e_k)}R^N)(df(e_k),\tau(f))V.\end{aligned}$$

Substituting these into (64), we have (59). $\boxed{QED}$

By the second variation formula, we derive the notion of stable 2-harmonic maps.

17 Definition. Let $f : M \to N$ be a 2-harmonic map of a compact Riemannian manifold M into any Riemannian manifold N. If the second variation of 2-energy is non-negative for every variation $\{f_t\}$ of f, i.e., the RHS of (59) is non-negative for every vector field V along f, f is said to be a *stable 2-harmonic map.*

By definition of 2-energy, any harmonic maps are stable 2-harmonic maps. This may also be seen as follows: since $\tau(f) = 0$, for a vector field V of any variation $\{f_t\}$ we have

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = \int_M \| -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i), V)df(e_i)\|^2 * 1 \geq 0.$$

18 Theorem. *Assume that M is a compact Riemannian manifold, and N is a Riemannian manifold with a positive constant sectional curvature $K > 0$. Then, there is no non-trivial stable 2-harmonic map satisfying the conservation law.*

Proof. Since N has constant curvature, $\nabla' R^N = 0$, so that (59) becomes

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) &= \int_M \| -\overline{\nabla}^*\overline{\nabla}V + R^N(df(e_i), V)df(e_i)\|^2 * 1 \\ &+ \int_M \langle V, R^N(\tau(f), V)\tau(f) + 2R^N(df(e_i), V)\overline{\nabla}_{e_k}\tau(f) \\ &\quad + 2R^N(df(e_i), \tau(f))\overline{\nabla}_{e_i}V\rangle * 1. \end{aligned} \tag{65}$$

Especially, if we take $V = \tau(f)$, then, the first term of the RHS of (65) and the first integrand of the second term vanish, so we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) &= 4\int_M \langle R^N(df(e_k), \tau(f))\overline{\nabla}_{e_k}\tau(f), \tau(f)\rangle * 1 \\ &= 4K \int_M \left[\langle df(e_k), \overline{\nabla}_{e_k}\tau(f)\rangle \|\tau(f)\|^2 \right. \\ &\quad \left. - \langle df(e_k), \tau(f)\rangle\langle\tau(f), \overline{\nabla}_{e_k}\tau(f)\rangle\right] * 1. \end{aligned} \tag{66}$$

Since f satisfies the conservation law, i.e., $-\langle\tau(f), df(X)\rangle = (\mathrm{div}S_f)(X) = 0$ for all $X \in \mathfrak{X}(M)$, we have

$$\langle df(e_k), \tau(f)\rangle = 0,$$

and

$$\begin{aligned}\langle df(e_k), \overline{\nabla}_{e_k}\tau(f)\rangle &= -\langle\overline{\nabla}_{e_k}df(e_k), \tau(f)\rangle + e_k\langle df(e_k), \tau(f)\rangle \\ &= -\|\tau(f)\|^2 - \langle df(\nabla_{e_k}e_k), \tau(f)\rangle \\ &= -\|\tau(f)\|^2. \end{aligned} \tag{67}$$

Substituting (67) into (66), we have

$$0 \leq \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) = -4K\int_M \|\tau(f)\|^4 * 1 \leq 0,$$

which implies that $\tau(f) \equiv 0$. $\boxed{QED}$

In order to apply the second variation formula, we take $N = \mathbb{C}P^n$.

19 Lemma. *Assume that $f : M \to \mathbb{C}P^n$ is a stable 2-harmonic map of a compact Riemannian manifold which satisfies the conservation law and $\|\tau(f)\|^2 > 3\sqrt{2e(f)}\,\|\overline{\nabla}\tau(f)\|$ pointwisely on M. Then, f is harmonic. Here, we denote $\|\overline{\nabla}\tau(f)\|^2 = \langle \overline{\nabla}_{e_k}\tau(f), \overline{\nabla}_{e_k}\tau(f)\rangle$.*

Proof. Assume that f satisfies all the assumption, but not harmonic. Since $\nabla' R^N = 0$, if we take $V = \tau(f)$, both the first term and the integrand of the second term of (65) vanish, and we use the explicit formula of the curvature tensor of $\mathbb{C}P^n$, (65) becomes as follows.

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\bigg| E_2(f_t) &= 4\int_M \langle R^N(df(e_k),\tau(f))\overline{\nabla}_{e_k}\tau(f),\tau(f)\rangle * 1\\
&= C\int_M \langle\langle df(e_k),\overline{\nabla}_{e_k}\tau(f)\rangle\tau(f) - \langle\tau(f),\overline{\nabla}_{e_k}\tau(f)\rangle df(e_k)\\
&+ \langle Jdf(e_k),\overline{\nabla}_{e_k}\tau(f)\rangle J\tau(f) - \langle J\tau(f),\overline{\nabla}_{e_k}\tau(f)\rangle Jdf(e_k)\\
&+ 2\langle Jdf(e_k),\tau(f)\rangle J\overline{\nabla}_{e_k}\tau(f),\tau(f)\rangle * 1,\end{aligned} \tag{68}$$

where C is a positive constant depending only on $\mathbb{C}P^n$. By (67) and $\langle J\tau(f),\tau(f)\rangle = 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\bigg| E_2(f_t) = C\int_M [&-\|\tau(f)\|^4\\
&+ 3\langle Jdf(e_k),\tau(f)\rangle\langle J\overline{\nabla}_{e_k}\tau(f),\tau(f)\rangle] * 1.\end{aligned} \tag{69}$$

For each k, by Schwarz inequality twice, we have

$$\begin{aligned}&\langle Jdf(e_k),\tau(f)\rangle\langle J\overline{\nabla}_{e_k}\tau(f),\tau(f)\rangle\\
&\le \sqrt{\langle Jdf(e_k),Jdf(e_k)\rangle}\|\tau(f)\|\sqrt{\langle J\overline{\nabla}_{e_k}\tau(f),J\overline{\nabla}_{e_k}\tau(f)\rangle}\|\tau(f)\|\\
&= \|\tau(f)\|^2\sqrt{\langle df(e_k,df(e_k)\rangle\langle\overline{\nabla}_{e_k}\tau(f),\overline{\nabla}_{e_k}\tau(f)\rangle}.\end{aligned}$$

By taking the sum over k, and by Schwarz inequality, we have

$$\begin{aligned}&\langle Jdf(e_k),\tau(f)\rangle\langle J\overline{\nabla}_{e_k}\tau(f),\tau(f)\rangle\\
&\le \|\tau(f)\|^2\sqrt{\langle df(e_i),df(e_i)\rangle\langle\overline{\nabla}_{e_j}\tau(f),\overline{\nabla}_{e_j}\tau(f)\rangle}\\
&= \sqrt{2e(f)}\|\tau(f)\|^2\|\overline{\nabla}\tau(f)\|.\end{aligned} \tag{70}$$

Substituting this into (69), we have

$$0 \le \frac{1}{2}\frac{d^2}{dt^2}\bigg| E_2(f_t) \le C\int_M \|\tau(f)\|^2\left(3\sqrt{2e(f)}\|\overline{\nabla}\| - \|\tau(f)\|^2\right) * 1$$

which is impossible if $\|\tau(f)\|^2 > 3\sqrt{2e(f)}\|\overline{\nabla}\tau(f)\|$. $\boxed{QED}$

20 Lemma. *Assume that $f : M \to N = \mathbb{C}P^n$ a 2-harmonic map from a compact Riemannian manifold into $\mathbb{C}P^n$ with constant holomorphic sectional curvature $C > 0$ which satisfies the conservation law and $\|\tau(f)\|^2 =$ constant. Then, it holds that*

$$\frac{C}{2} e(f)\|\tau(f)\|^2 \le \|\overline{\nabla}\tau(f)\|^2 \le 2Ce(f)\|\tau(f)\|^2. \tag{71}$$

Proof. Since f is 2-harmonic, we can still use the equality in (37), so that

$$0 = \frac{1}{2}\Delta\|\tau(f)\|^2 = \|\overline{\nabla}\tau(f)\|^2 - \langle R^N(df(e_i), \tau(f))df(e_i), \tau(f)\rangle. \tag{72}$$

We denote by $\mathrm{Riem}^N(df(e_i) \wedge \tau(f))$, the sectional curvature through $df(e_i)$ and $\tau(f)$. Since this plane does not degenerate, and f satisfies the conservation law, for each i,

$$\begin{aligned} &\langle R^N(df(e_i), \tau(f))df(e_i), \tau(f)\rangle \\ &= \mathrm{Riem}^N(df(e_i) \wedge \tau(f)) \cdot \langle df(e_i), df(e_i)\rangle \, \|\tau(f)\|^2. \end{aligned} \tag{73}$$

Recall that the sectional curvature of $\mathbb{C}P^n$ satisfies

$$\frac{C}{4} \le \mathrm{Riem}^N \le C, \tag{74}$$

so that by (73), (74), we have

$$\begin{aligned} \frac{C}{2} e(f)\|\tau(f)\|^2 &\le \langle R^N(df(e_i), \tau(f))df(e_i), \tau(f)\rangle \\ &\le 2Ce(f)\|\tau(f)\|^2. \end{aligned} \tag{75}$$

Thus, we have (71). $\boxed{QED}$

21 Theorem. *Let $f : M \to \mathbb{C}P^n$ a stable 2-harmonic map from a compact Riemannian manifolds M into $\mathbb{C}P^n$ with constant holomorphic sectional curvature $C > 0$, which satisfies the conservation law, and $\|\tau(f)\|^2 =$ constant. If the density function of f satisfies*

$$e(f) < \frac{\|\tau(f)\|}{6\sqrt{C}}, \tag{76}$$

then f is harmonic.

Proof. Assume that there exists such a stable 2-harmonic map but not harmonic. By Lemma 9, there exists a point $p \in M$ at which

$$0 < \|\tau(f)\|^2 \le 3\sqrt{2e(f)}\|\overline{\nabla}\tau(f)\|.$$

By Lemma 9, it holds that, at this point,

$$\|\tau(f)\|^2 \leq 3\sqrt{2e(f)}\|\overline{\nabla}\tau(f)\| \leq 6\sqrt{C}e(f)\|\tau(f)\|.$$

Then, at this point,

$$0 \leq \|\tau(f)\|(6\sqrt{C}e(f) - \|\tau(f)\|).$$

Since $\|\tau(f)\| > 0$ at p, we have $6\sqrt{C}e(f) - \|\tau(f)\| \geq 0$ at p which contradicts the assumption (76). $\boxed{QED}$

22 Corollary. *Assume that $f : M \to \mathbb{C}P^n$ is a 2-harmonic isometric immersion from a m-dimensional compact Riemannian manifold M into $\mathbb{C}P^n$ with constant holomorphic sectional curvature $C > 0$ whose $\|\tau(f)\|$ is constant and satisfies $\|\tau(f)\| > 3\sqrt{C}m$. Then f can not be stable.*

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